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THE INITIAL STAGE IN THE EVOLUTION OF DISPLACEMENT FRONTS IN THE NON-LINEAR FILTRATION PROBLEM†

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A non-linear filtration model is considered. The initial stage in the evolution of the displacement front is studied and the initial conditions are found under which the front maintains a stable form for a long time. Concentration and pressure profiles in a direction normal to the front are obtained. It is found that the concentration varies comparatively rapidly here while the pressure varies extremely slowly. © 1997 Elsevier Science Ltd. All rights reserved.

1. DESCRIPTION OF THE PROBLEM AND THE CHOICE OF MODEL

We consider the two-dimensional unsteady isothermal problem of two-component filtration in a homogeneous and isotropic porous medium without phase transitions and we will model the displacement of one liquid or gas by another liquid or gas.

Suppose that the displacing substance is characterized by a partial density ρ_1, a density of the pure substance ρ_{10} and a viscosity μ_1. We denote its concentration in the mixture by c_1. The corresponding characteristics of the substance which is being displaced are ρ_2, ρ_{20}, μ_2, c_2. In an important practical situation $\mu_1 < \mu_2$, $\rho_1 > \rho_2$. We assume that the mixture density $\rho = \rho_1 + \rho_2$ and the concentration of the substance which is being displaced $c = c_2 = \rho_2(\rho_1 + \rho_2)^{-1}$ are the basic quantities which are to be found and that the mixture viscosity $\tilde{\mu} = \tilde{\mu}(c)$ which varies between the values of μ_1 and μ_2 depends solely on the concentration c. It is further assumed that $\tilde{\mu}'(c) \geqslant 0$ everywhere and that the pressure p depends solely on the total density ρ.

The system of equations

$$m\tilde{\rho}_t = \operatorname{div}\left(\frac{kp_\rho}{\tilde{\mu}(c)}\tilde{\rho}\nabla\tilde{\rho}\right), \quad mc_t = \frac{kp_\rho}{\tilde{\mu}(c)}(\nabla\tilde{\rho}, \nabla c) + D\Delta c \tag{1.1}$$

can be written for the above-mentioned unknown functions of the coordinates and time. The coefficients introduced here are: the porosity m, the permeability of the porous medium k, the derivative of the pressure with respect to the density p_ρ, and the coefficient of diffusion D, which it is natural to assume to be known. The function $\tilde{\mu}(c)$ is also assumed to be known. Equation (1.1) is a consequence of the usual equations of continuity, diffusion and Darcy's law [1, 2].

We next make use of the concept of a displacement front which is a closed curve l which expands with time and in the neighbourhood of which the most rapid change in the concentration c occurs. This definition of a displacement front is justified if the dimensionless parameter, which includes the diffusion coefficient, is small (Section 2). A pressure well is located within l and, in its neighbourhood, $\rho \approx \rho_{10}$, $c \approx 0$. Outside l, there is an extraction well and, here, $\rho \approx \rho_{20}$, $c \approx 1$.

The first complication in the non-linear (2 + 1)-dimensional problem under discussion is the fact that, to solve it, it is not only necessary to find the unknown functions $\tilde{\rho}$ and c but also the unknown curve l. A second difficulty is associated with the observed instability of the displacement front which manifests itself in the invasion of the displacing substance into the external (oil and gas) zone in the form of narrow viscous fingers which are also called Saffman–Taylor fingers [3]. The principal aim of this paper is to investigate the evolution of the displacement front during its initial stage when the formation of viscous fingers only commences.

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2. CONVERSION TO DIMENSIONLESS QUANTITIES AND SMALL PARAMETERS

We now introduce certain mean (effective) values for the density and viscosity as well as the scales of length L and of time T by which the space and time coordinates are divided while retaining the previous notation both for these quantities and for the operations div, ∇ and Δ. For the dimensionless functions $\rho = \tilde{\rho}/\rho_0$, c, $\mu(c) = \tilde{\mu}(c)/\mu_0$, instead of (1.1), we obtain the system of equations

$$\rho_t = A\,\mathrm{div}\frac{\rho\nabla\rho}{\mu(c)}, \quad c_t = A\frac{\nabla\rho, \nabla c}{\mu(c)} + B\Delta c, \tag{2.1}$$

$$A = \frac{k p_\rho \rho_0}{m\mu_0}\frac{T}{L^2}, \quad B = \frac{D}{m}\frac{T}{L^2}$$

The magnitude of the dimensionless and constant coefficients A and B can be estimated. Two versions are possible under real conditions

$$1.\ A = A_0, B = \varepsilon^2 B_0; \quad 2.\ A = \varepsilon A_0, B = \varepsilon^2 B_0$$

in which A_0 and B_0 are of the order of unity and $\varepsilon \sim 10^{-5}$. These versions are the same in the principal scheme. We shall subsequently consider the first version, that is, the system of equations

$$\rho_t = \mathrm{div}(\mu^{-1}\rho\nabla\rho), \quad \rho|_{t=0} = \rho_0(\mathbf{x}); \quad c_t = \mu^{-1}(\nabla\rho, \nabla c), \quad c|_{t=0} = c_0(\mathbf{x}) \tag{2.2}$$

in which we put $A_0 = 1$. Otherwise, it can be converted to the variable $\tau = A_0 t$. It is clear that flow in a weakly inhomogeneous porous medium can also be described within the framework of the system of equations (2.2).

Such singularly perturbed systems can be investigated using two different methods, which give equivalent results.

The first method involves the substitution of a formulation of the Witham type

$$\rho = R(\delta^{-1}S(x,t,\delta), x, t, \delta), \quad c = C(\delta^{-1}S(x,t,\delta), x, t, \delta)$$

where S is an unknown phase and δ is a small parameter associated with ε. In the case of C and R, a so-called calibration problem arises which often has a self-similar solution (which depends on $x - vt$, for example). A certain equation, which also describes the propagation of the wave front l, also arises naturally in the case of S.

The second method is associated with the ideas of a boundary layer. In the neighbourhood of the curve l, we introduce the dimensionless length of the normal $|n|$ to l and the length of the arc s along l. By postulating that l is closed, smooth and that it does not intersect itself, we calculate that $n > 0$ outside the domain bounded by l and that $n < 0$ within this domain. We also introduce the fast variable $\nu = \delta^{-1}n$. After this, a calibration equation also arises (which is analogous to that mentioned above). A recurrence procedure then enables us to describe the geometrical characteristics to the front l and the law for its evolution with time.

The scheme which has been described operates successfully in a number of important problems which are described by second-order quasilinear equations. However, a number of fundamental difficulties also arise here which intrinsically distinguish system (2.2) from the systems considered earlier.

3. "A PRIORI" ESTIMATES AND ESTIMATION OF THE LOCAL VELOCITY OF MOTION OF THE DISPLACEMENT FRONT

We know from general theory [4, 5] that in order to prove the existence of solutions of system (2.2) for all $t > 0$, it is sufficient to obtain certain a priori estimates. These estimates, as well as certain information on the evolution of the solutions, can be obtained by the method of upper and lower functions [5].

It is natural to assume that the initial values $\rho_0(\mathbf{x})$ and $c_0(\mathbf{x})$ of the solutions of (2.2) satisfy the conditions

$$\rho_{20} < \rho_0(\mathbf{x}) < \rho_{10}, \quad 0 < c_0(\mathbf{x}) < 1 \tag{3.1}$$

It can be seen that the constants ρ_{20}, 0 and ρ_{10}, 1 give the lower and upper solutions. Consequently, we have the a priori estimates

$$\rho_{20} \leqslant \rho(\mathbf{x},t) < \rho_{10}, \quad 0 \leqslant c(\mathbf{x},t) \leqslant 1 \tag{3.2}$$

which guarantee the existence of a solution.

For the one-dimensional version

$$\rho_t = (\rho\rho_x\mu^{-1}(c))_x, \quad c_t = \mu^{-1}(c)c_x\rho_x + \varepsilon^2 c_{xx} \tag{3.3}$$

additional information can be obtained. We assume that, at the initial instant $t > 0$, the function $c_0(x) = c(x, 0)$ is monotonic and $c_0'(x) > 0$. We will prove that a similar property also holds for all $t > 0$. We differentiate the second equation of (3.3) with respect to x and obtain an equation containing $u = c_x$. For this equation $u \equiv 0$ is the lower solution. Then [5]

$$c_x(x, t) \geqslant 0 \tag{3.4}$$

Similarly, the supposition that $\rho_0'(x) < 0$ implies the inequality

$$\rho_x(x,t) \leqslant 0 \tag{3.5}$$

The estimates for ρ_x and c_x which have been obtained enable us to estimate the function $\rho(x, t)$ from below. It turns out that this function majorizes the function $Z^{1/2}(x, t)$ which satisfies the equation $Z_t = \mu Z^{1/2} Z_{xx}$.

The solutions of this equation behave in roughly the same way as solutions of the standard heat-conduction equation. In particular, their spreading with respect to x as t increases follows from the equality

$$\int_{-\infty}^{\infty} Z_t^2 (\mu Z^{1/2})^{-1} dx = -\frac{d}{dt}\int_{-\infty}^{\infty} Z_x^2 dx$$

It is then possible to estimate $|\rho_x|$ over a certain range

$$0 \leqslant t \leqslant T_\varepsilon \tag{3.6}$$

of the initial evolution of the displacement front when $|c_x| \leqslant c_1\varepsilon^{-1}$, $|c_{xx}| \leqslant c_2\varepsilon^{-2}$. As the lower function for ρ_x, we take the function $-a(t)\mu(x, t)$. It can be shown that the function $a(t)$ can be expressed in the form

$$a(t) = \sup_x |\rho_{0x}| \mu(x,t) \exp(c_2 t) \tag{3.7}$$

Here c_2 is a constant which arises when estimating $|c_{xx}|$ in the range (3.6)

$$\sup_x |c_{xx}(x,t)| < c_2\varepsilon^{-2}$$

Finally, using (3.5), it is possible to construct upper and lower limits for the function $c(x, t)$

$$\tilde{c}(x - At, t) \leqslant c(x,t) \leqslant \tilde{c}(x + At, t), \quad A = \sup_{0<t<T_\varepsilon} a(t) \tag{3.8}$$

where $\tilde{c}$ is the solution of the heat-conduction equation

$$\tilde{c}_t = \varepsilon^2 \tilde{c}_{xx}, \quad \tilde{c}|_{t=0} = c_0(x) \tag{3.9}$$

Inequalities (3.8) enable one to estimate the velocity of the front motion during the initial stage of the process. Actually, the front curvature can increase in the range (3.6) but its rate of displacement does not exceed the constant A, which limits the magnitude of $\mu^{-1}(c_0)|\nabla\rho_0|$ from below.

The estimate which has been obtained leads to the following practical recommendation. The instability of a displacement front is smaller, the smaller the derivative of the expression $\rho_{0n}\mu^{-1}(c_0)$ along l, that is, the derivative with respect to the variable s.

The physical mechanism by which instability of the front shape occurs can be described using the above estimates and calculations. The subsequent arguments are not completely rigorous since they are partially based on the estimates obtained above, which have only been proved in the one-dimensional case. However, these estimates are quite convincing in the case of a quasiplane front, that is, a front with a small initial curvature.

So, we shall assume that, at the initial instant, the vector

$$\mathbf{V}(\mathbf{x}, t) = \mu^{-1}\nabla\rho$$

varies extremely smoothly such that $|\nabla\mathbf{V}| < \delta$, $|\nabla^2\mathbf{V}| < \delta \ll 1$. So long as the magnitude of $\varepsilon^2|\Delta c|$ is small, this vector, as previously, varies weakly with respect to $\mathbf{x}$ and barely changes as t increases from its initial value, which is proved in a similar manner to the derivation of the lower function for ρ_x. Equation (2.6) is therefore well-approximated by the equation

$$c_t = \mathbf{V}(\delta\mathbf{x}, \delta t)\nabla c$$

or, when account is taken of the assumption of a quasiplane front, by the equation $c_t = V(\delta x, \delta t)c_n$. With the initial condition $c|_{t=0} = c_0(\mathbf{x})$, the corresponding solution has the form $c(\mathbf{x}, t) = c_0(\mathbf{x} + \mathbf{V}t)$. This means that each segment of the front propagates with its own local velocity $\mathbf{V}$, which is defined by the initial value of $\mu^{-1}\nabla\rho$. Hence, here we have the conventional pattern of kinematic instability when different segments of a front move at a different velocity.

As $|\nabla\mathbf{V}|$ and $|\nabla_c|$ increase, the pattern becomes more complex. It is necessary to use other methods to describe the behaviour of the solutions at times when $|\nabla_c| \sim \varepsilon^{-1}$. This is done in Section 5 with the additional assumption that the front curvature is small.

4. THE STANDARD PROBLEM

We will attempt to find a solution $\rho = R(z)$, $c = C(z)$, $z = x - vt$ of the one-dimensional equations (3.3) which is self-similar in the highest order. Here $\mu(c) = M(z)$. For brevity, let $\rho_i \equiv \rho_{i0}$.

The functions R and C must satisfy the boundary conditions

$$R(z_+) = \rho_2,\quad R(z_-) = \rho_1,\quad C(z_+) = 1,\quad C(z_-) = 0, \tag{4.1}$$

where for a problem in an infinite interval $z_\pm = \pm\infty$.

After substitution and integration, we have

$$-vR + Kv = RR'M^{-1},\ K = \text{const} \tag{4.2}$$

$$-C'(v + R'M^{-1}) = \varepsilon^2 C'' \tag{4.3}$$

After using the boundary condition, Eq. (4.2) yields

$$v\int_{z_-}^{z_+} M(z)dz = S(K) \equiv \rho_1 - \rho_2 + K\ln\frac{\rho_1 - K}{\rho_2 - K} \tag{4.4}$$

It is clear that there is no solution of the problem being considered over an infinite range of variation of z since the modulus of the right-hand side of (4.4) is no less than $(z_+ - z_-)|\, v\, |\mu_1$. However, using (4.4), it is possible to prove the existence of solutions of the two-point boundary-value problem (4.1)–(4.3) over a large, but finite, interval when $z_\pm = \pm\delta^{-1}$, where δ is a small parameter.

Suppose that the function R is expressed in the form

$$R = R_1(-vU(z), K),\quad \rho_2 \leqslant R \leqslant \rho_1;\quad U(z) = \int_{z_-}^{z} M(C(z'))dz' \tag{4.5}$$

An expression for R_1 can be found from the equation $vU = \rho_1 - R_1 + K\ln(\rho_1 - K) - K\ln(R_1 - K)$. It is a completely continuous functional of $C(z)$, if $C(z)$ is a continuous function. Suppose that

$$\Phi(z,\delta,\varepsilon,K) = \upsilon K\varepsilon^{-2}\int_z^{\delta^{-1}} \frac{dz'}{R(z')}, \quad \langle\mu\rangle = 2\delta\int_{-\delta^{-1}}^{\delta^{-1}} M(z)dz$$

The constant v is defined by the relation $v = 2\delta S(K)\langle\mu\rangle^{-1}$ which follows from (4.4). Substitution of (4.5) into (4.3) leads to the equality

$$C(z) = (c_2 - c_1)[\exp\Phi(z,\delta,\varepsilon,K) - E_-](E_+ - E_-)^{-1} + c_1 \equiv T[C(\cdot)](z) \tag{4.6}$$

$$c_{2,1} \equiv C(\pm\delta^{-1}), \quad E_\pm \equiv \exp\Phi(\pm\delta^{-1},\delta,\varepsilon,K)$$

Here, the phase Φ is a non-linear, completely continuous functional in the set of functions $C(z)$ which are continuous in $(-\delta^{-1}, \delta^{-1})$ such that $c_1 \leqslant C(z) \leqslant c_2$. According to (4.6), T is such a functional and maps the above-mentioned set of functions into themselves. Consequently, by Schauder's principle, a solution of Eq. (4.6) exists. Then, for each K, such that $S(K) > 0$, a solution of the boundary-value problem (4.1)–(4.3) exists with a certain positive value of the velocity v.

The solution obtained can be used in the following manner for small $\delta \ll \varepsilon^2$. In this case, we have

$$C'(-\delta^{-1}) = O(e^{-c\varepsilon^2}), \quad C'(\delta^{-1}) = O(\delta\varepsilon^{-2}), \quad R'(\pm\delta^{-1}) = O(\delta)$$

Consequently, if we continue the solution beyond the limits of the range $(-\delta^{-1}, \delta^{-1})$ by putting

$$C = c_1, \quad R = \rho_1 \ (z < -\delta^{-1}), \quad C = c_2, \quad R = \rho_2 \ (z > \delta^{-1})$$

we obtain the generalized solution of class $W^{2,1}$ which satisfies the system of equations with a small residual (the residual is a functional $L_{\varepsilon,\delta}$ in the class of Schwartz functions with a norm not exceeding $\delta\varepsilon^{-2}$). A similar idea has been used in [6].

We shall now consider the behaviour of the solution $C(z)$, $R(z)$ in detail. Formula (4.6) shows that the concentration c changes sharply in the neighbourhood of the front. The steepness of the graph of this function increases as ε decreases. This is due to the fact that Eq. (4.3) contains a small parameter. At the same time, the density ρ and the pressure p change comparatively slowly in the neighbourhood of the displacement front while the pressure gradient is also appreciable at a large distance from the displacement front.

5. CRITERIA FOR THE STABILITY OF A FRONT WITH A SMALL CURVATURE

The problem of the evolution of a front with a small curvature, when $|\nabla\rho|$ and $|\nabla_c|$ are small and $|(\nabla\rho, \nabla_c)|$ is much greater than $\varepsilon^2\Delta c$, has been considered in Section 3. It might be expected that, as a result of evolution with time, solutions are formed for which the term $\varepsilon^2\Delta c$ can now no longer be neglected. The problem as to how the normal velocity and the curvature of such fronts change with time in this case is of interest. This problem has been solved for the simplest equations and systems (see [7–10]).

In a homogeneous medium (when the coefficients of the equations are independent of x and t), for the front velocity v in the direction normal to the front, we have

$$\upsilon = -\alpha\varkappa + \upsilon_0 \tag{5.1}$$

where $\varkappa$ is the front curvature (the mean curvature in the multidimensional case) and v_0 is a constant which is determined from the solution of the standard problem. A certain procedure can be proposed for calculating the constant α.

In the case of the equations from [8, 9], $\alpha > 0$ always. As has been shown by Kuramoto [10], it is possible that $\alpha < 0$ for certain other systems. For such systems, there is diffusional instability of the front, and a front which initially has a small curvature can turn out to be strongly bent after a short time.

The principal physical effect, that is, whether the front is stable or not, is completely determined by the sign of α.

In the problem under consideration, which is significantly more complex than those which have been studied earlier, it is difficult to create a regular procedure for determining the corrections. We shall use another method which enables us to determine the sign of α. It can be checked that this method also gives the correct answer in the cases which have been studied previously.

Suppose that the front is a smooth plane curve l, without any self-intersections and with a very small curvature $\varkappa$, $|\varkappa| \ll \delta \ll \varepsilon^2$, where δ is the small parameter used in Section 4. On introducing the standard coordinates (n, s), where n is the length of the normal to the front and s is the length of an arc along it, we obtain the system of equations

$$-v\,\rho_n = \left(\frac{\rho\rho_n}{\mu(c)}\right)_n + \varkappa(s)\left[\frac{\rho\rho_n}{\mu} - n\left(\frac{\rho\rho_n}{\mu}\right)_n\right] \tag{5.2}$$

$$-v\,c_n = \rho_n c_n \mu^{-1}(c) + \varepsilon^2 c_{nn} \tag{5.3}$$

We assume that the term with $\varkappa$ in (5.2) is of the nature of a correction term in accordance with the condition $|\varkappa| \ll \delta$ and since $v \sim \delta$. The functions ρ and c can therefore be determined using the formulae in Section 4.

We integrate Eq. (5.2) with respect to n in the limits from $-\delta^{-1}$ to δ^{-1} and, after this, we find that

$$\alpha = 2\int_{-1/\delta}^{1/\delta} \frac{\rho\rho_n}{\mu}\,dn - n\frac{\rho\rho_n}{\mu}\bigg|_{-1/\delta}^{1/\delta} \tag{5.4}$$

Here it follows that one should make the substitution $\rho = R(n)$. In the problems from [8, 9] which have been studied, the term outside the integral always vanished and the sign of α was easily determined from a consideration of the monotonicity of ρ with respect to n. A certain accuracy is now required since the two terms in (5.4) are of the same order.

We use (4.2), from which, after some reduction, we have

$$\alpha = \frac{v}{4\delta}\left(\frac{\rho_{10}+\rho_{20}}{2} - \langle R\rangle\right), \quad \langle R\rangle = 2\delta\int_{-\delta^{-1}}^{\delta^{-1}} R\,dn \tag{5.5}$$

Hence, the sign of α depends on what is the greater, the mean of R over all n or the means of the boundary values ρ_{10} and ρ_{20}. Stable propagation occurs in the first case while the front disintegrates in the second case.

To analyse the magnitude of α, numerical methods were employed directly for the initial partial differential equations in the spatially one-dimensional case when $0 \leqslant x \leqslant L$, $\rho(0) = \rho_1 > \rho_2 = \rho(L)$, $c(0) = 0$, $c(L) = 1$. If the function $\rho(x)$ were to be linear, we would obtain that $\alpha = 0$. However, the calculations show that the function $\rho(x)$ is convex upwards although only very, very weakly. The magnitude of α is therefore very small in absolute magnitude but it is negative, that is, the front is weakly stable When account is taken of the fact that the process is not one-dimensional, instability of the displacement front can occur during the very first stage of the process.

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